Non-Parallel Yield Curve Shifts and Stochastic Immunization

A new and flexible paradigm for managing yield curve risk.

Robert R. Reitano

General price function models reflecting multivariate yield curve specifications make it possible to define a variety of duration and convexity measures that greatly improve the understanding of yield curve shift risk (see Ho [1990, 1992] and Reitano [1989, 1990a, 1991a, 1991b, 1992b]). Traditional immunization theories also generalize to these yield curve models (Reitano [1990b, 1991c, 1992a]).

In general, we can conclude that the larger the class of yield curve shifts against which we seek immunization, the more restrictive the necessary immunization conditions become. Conversely, the smaller the class, the more likely the strategy will fail.

For example, immunization against parallel shifts requires a constraint on the modified durations of assets and liabilities, and also a constraint on convexities, although this latter restriction is generally far less important in practice. Analogously, against any other one-parameter model for yield curve shifts, we need identical restrictions on so-called directional durations and directional convexities, where the direction vector reflects the shape of the shift assumed. These strategies are easy to implement, yet are likely to fail when an unanticipated shift occurs.

Using a principal component or other analysis, and choosing several yield curve shifts to immunize against, as in Litterman and Scheinkman [1991], we now need a like number of directional duration and directional convexity constraints, where the direction vectors

ROBERT R. REITANO is second vice president of the investment policy and research department at John Hancock Mutual Life Insurance Company in Boston (MA 02117).

reflect the "shape" of the principal components or shifts chosen.[1] The immunizing conditions necessary to immunize against all yield curve shifts allowable within a given yield curve model put restrictions on the asset and liability partial durations and partial convexities. As the yield curve model is refined further, these partial duration constraints simply push the portfolios toward cash matching in a fixed cash flow application. When embedded options exist, generalized cash matching occurs in the sense that both portfolios will have identical behaviors with respect to arbitrary "infinitesimal" yield curve shifts.

There are clearly practical shortcomings of implementing such severe restrictions. Consider also that one may be inadvertently immunizing against even infeasible yield curve shifts and shifts too unlikely to worry about.

This article develops a new framework for full yield curve immunization. Rather than seek immunization in the classic sense of eliminating downside risk, we seek immunization in the stochastic sense of risk minimization. The risk measure used is a weighted-average of the portfolio variance, as used in Markowitz [1959], and a measure of worst case yield curve risk. We provide explicit formulas for the solution of the risk minimization problem that can be made to satisfy various constraints on one or several directional durations and the expected yield curve return, as well as other constraints reflecting the specific assets available for trading.

The theory is presented through the detailed analysis of an example introduced in Reitano [1990a, 1991a, 1992b].[2] For a review of other stochastic approaches to duration matching and dedication, see Hiller and Schaak [1990].

SETUP OF THE GENERAL PROBLEM: RISK

We denote by $P(\bar{i})$ a price function defined on a yield curve vector, $\bar{i} = (i_1, ..., i_m)$, which describes the term structure through the identification of m yield points, which I call yield curve drivers, and Ho [1992] calls key rates. Yields at other maturities are assumed to depend on these m identified yields in a formulaic way, such as via interpolation.

The current yield curve vector is denoted $\bar{i}_0$, and the yield curve shift model is denoted:

$$\bar{i}_0 \rightarrow \bar{i}_0 + \overline{\Delta i} \tag{1}$$

where $\overline{\Delta i} = (\Delta i_1, ..., \Delta i_m)$ denotes the vector of quantities by which each yield point shifts during a given and fixed time period. During this period, $\overline{\Delta i}$ is interpreted as a random vector.

In practice, $P(\bar{i})$ may denote any price function, but for immunization applications, it will usually denote the surplus price function, $S(\bar{i})$, or forward surplus price function at time t, $S_t(\bar{i})$ (see Reitano [1992a, 1993]). For yield curve shift analyses over shorter periods, the distribution of $\overline{\Delta i}$ may reflect a historical analysis of actual yield curve shifts or a theoretical model based on such an analysis, with the theory below then applied to $S(\bar{i})$. For longer time periods, for which simple yield curve differences distort the true evolution of yields due to time drift, it is better to base the analysis of historical shifts on a predictor yield curve interpretation, and apply the results below to $S_t(\bar{i})$.

As developed in greater detail in Reitano [1995], the yield curve shift, $\overline{\Delta i}$, is then defined as the shift from the initial yield curve, $\bar{i}_0$, to the predictor yield curve of the end-of-period curve, $\bar{i}^P$. Predictor yield curves are denominated in beginning-of-period time units, thereby allowing a simple yet meaningful subtraction: $\overline{\Delta i} = \bar{i}^P - \bar{i}_0$.

As detailed in Reitano [1993], the ratio function, $P(\bar{i}_0 + \overline{\Delta i})/P(\bar{i}_0)$ can be linearly approximated by $R(\overline{\Delta i})$:

$$\begin{aligned} R(\overline{\Delta i}) &= 1 - \sum D_j(\bar{i}_0)\,\Delta i_j \\ &= 1 - \overline{D}(\bar{i}_0) \cdot \overline{\Delta i} \end{aligned} \tag{2}$$

where the $D_j(\bar{i}_0)$ are the partial durations of $P(\bar{i})$ evaluated on $\bar{i}_0$. The second expression in (2) reflects the dot or inner product notation for the vector product of $\overline{D}(\bar{i}_0) = [D_1(\bar{i}_0), ..., D_m(\bar{i}_0)]$, called the total duration vector, and $\overline{\Delta i}$, the vector shift. We formally assume $P(\bar{i}_0) \neq 0$, although the theory can readily be adapted to circumvent this restriction.

We now define a general risk measure for the price function $P(\bar{i})$. Following Markowitz [1959], it is natural to consider the variance of $R(\overline{\Delta i})$, since this approximates the total return risk as measured by

$P(\bar{i}_0 + \overline{\Delta i})/P(\bar{i}_0)$. In addition, one may wish to limit outlier risk from the random vector, $\overline{\Delta i}$, which may not be adequately reflected in the variance measure because of a low implied weighting. To do this, we note that the absolute difference between $R(\overline{\Delta i})$ and its expected value is bounded by the product of the length of the total duration vector, denoted $|\overline{D}(\bar{i}_0)|$, and the length of the vector shift less its expected value, $|\overline{\Delta i} - E(\overline{\Delta i})|$. Consequently, if the length of $\overline{D}(\bar{i}_0)$ is made small, outlier risk to $R(\overline{\Delta i})$ is made small, so $|\overline{D}(\bar{i}_0)|$ can be interpreted as a worst case risk measure.

In light of these comments, we define a risk measure, RM(w), with weighting parameter w, as the weighted-average of these two risk measures:

$$RM(w) = w\,\mathrm{Var}[R(\overline{\Delta i})] + (1 - w)\,|\overline{D}(\bar{i}_0)|^2 \tag{3}$$

where the weighting parameter is chosen: $0 \le w \le 1$. Note that in (3) the length of $\overline{D}(\bar{i}_0)$ is reflected to the square so that it is of second-order as is the variance function.

This risk measure can be compactly expressed as a matrix product:

$$RM(w) = \overline{D}\,\overline{K}_w\overline{D}^T \tag{4}$$

where $\overline{D}$ denotes the total duration vector, which is by convention identified with a row matrix, $\overline{D}^T$ is its column matrix transpose, and $\overline{K}_w$ is the weighted-average of $\overline{K}$, the covariance matrix of $\overline{\Delta i}$, and the identity matrix, $\bar{I}$:

$$\overline{K}_w = w\overline{K} + (1 - w)\,\bar{I} \tag{5}$$

Of course, when $w = 1$ in (5), the risk measure in (4) reduces to $\mathrm{Var}[R(\Delta i)]$ by (3). Also, while in theory we require only that $0 \le w \le 1$, in practice we must choose w very close to 1, or the worst case risk term will dominate because of the difference in scale between the units of $\overline{K}$ and those of $\bar{I}$ (see the example below).

For our purposes, $\overline{K}_w$ has the mathematically convenient property of positive semidefiniteness, which we can assume to be in fact positive definiteness by a change in the yield basis. By positive definiteness is meant that RM(w) in (4) can be 0 only if $\overline{D}$ is the zero vector; otherwise, it is strictly positive. Geometrically this means that for any fixed value of $c > 0$, the set of total duration vectors defined by $RM(w) = c$ forms an ellipsoid in m-dimensional space.

SETUP OF THE GENERAL PROBLEM: CONSTRAINTS

Our goal is to minimize risk: RM(w). Of course, this is a trivial problem without further restriction on $\overline{D}(\bar{i}_0)$, because as noted above, the minimum of RM(w) is 0, and this is achieved exactly when $\overline{D}(\bar{i}_0) = \bar{0}$, the zero vector. If $P(\bar{i})$ reflects a surplus portfolio, this condition implies that assets and liabilities are dollar partial duration-matched. Consequently, the only question remaining is how to trade within the given asset portfolio to achieve this target total duration vector.

As it turns out, restrictions of interest on $\overline{D}(\bar{i}_0)$ involve restrictions on the dot product: $\overline{D} \cdot \overline{N}$, for various values of the vector $\overline{N}$. For example:

1. Directional durations, $D_N(\bar{i}_0)$, may be restricted in one or several directions, because: $D_N(\bar{i}_0) = \overline{D} \cdot \overline{N}$. For example, $\overline{N} = (1, 1, \ldots, 1)$ can be used to restrict traditional duration.
2. Expected period return, r′, also provides a restriction of this type since:

$$E[R(\overline{\Delta i})] = 1 + r' = 1 - \overline{D} \cdot \overline{N} \tag{6}$$

where $\overline{N} = E[\overline{\Delta i}]$ is the expected yield curve shift.
3. Asset trading set limitations can also be reflected this way with a little thought. Assume we have a specified collection of assets from which we wish to trade, with the logical constraint of cash neutrality: that total sales equal total purchases. This collection may or may not place a restriction on the total duration vectors that are achievable by trading in the given portfolio.

To find out, we form the matrix, $\overline{A}$, with column vectors equal to the total duration vectors: $\overline{D}_j - \overline{D}_n$, for $j = 1, \ldots, n-1$. Here, we assume that there are n assets, with $\overline{D}_j$ denoting the corresponding total duration vectors. We can choose any asset for $\overline{D}_n$; the purpose of the subtraction is to assure a cash-neutral trade.

We then solve the system of equations:

$$\overline{A}^T\overline{N} = \overline{0} \tag{7}$$

and look for the maximum number of linearly independent solutions. That is, we seek a basis for the null space of $\overline{A}^T$. Of course, (7) may have no non-trivial solutions (i.e., solutions other than $\overline{N} = \overline{0}$).

The given trading set of n assets then introduces the following constraints on feasible total duration vectors, $\overline{D}$:

$$\overline{D} \cdot \overline{N}_j = \overline{D}(i_0) \cdot \overline{N}_j \tag{8}$$

where $\overline{N}_j$ are the independent solutions to (7), and $\overline{D}(i_0)$ is the initial total duration vector. That is, (8) states that, given these n assets, and the requirement of a cash-neutral trade, one cannot change the directional durations of the original portfolio in the $\overline{N}_j$ directions.

Of course, if (7) has no non-zero solutions, which will happen when there are enough assets, no restrictions are introduced in (8).

SOLUTION OF THE GENERAL PROBLEM

We now present an explicit solution to a constrained minimization problem assuming only that the constraint vectors, $\overline{N}_j$, are consistent, i.e., linearly independent (see Reitano [1993] or Martin et al. [1988] for a derivation):

$$\text{Minimize: } \overline{D}\overline{K}_w\overline{D}^T$$

$$\text{Subject to: } \overline{D} \cdot \overline{N}_j = r_j, \; j = 1, \ldots, p \tag{9}$$

To express the solution compactly, we denote by $\overline{B}$ the matrix with the $\overline{N}_j$ vectors as columns, and by $\overline{r}$ the column vector of r_j values. Consequently, the p-constraints in (9) can be expressed: $\overline{D}\overline{B} = \overline{r}^T$, where we recall that total duration vectors are identified with row matrixes.

Denoting by $\overline{D}_0$ the solution to (9), we then have:

$$\overline{D}_0^T = \overline{K}_w^{-1}\,\overline{B}\,(\overline{B}^T\overline{K}_w^{-1}\overline{B})^{-1}\overline{r} \tag{10A}$$

and the value of the risk measure, $RM_0(w)$, is given in (4) by:

$$RM_0(w) = \overline{r}^T\,(\overline{B}^T\overline{K}_w^{-1}\overline{B})^{-1}\overline{r} \tag{10B}$$

While Equations (10A) and (10B) may look imposing, these matrix calculations are easily performed in a variety of computer software from spreadsheets to APL.

Of course, Equation (10A) provides the risk-minimizing total duration vector for the portfolio that satisfies the various constraints of interest, while (10B) gives the corresponding value for the risk measure defined in (3). One interesting consequence of expression (10B) is that one can achieve zero risk only if $\overline{r} = \overline{0}$, the zero vector. This is because $\overline{B}^T\overline{K}_w^{-1}\overline{B}$, and hence its inverse, are also positive definite.

To calculate the necessary cash-neutral trade from the given asset set is now easy. Let $\overline{a}' = (a_1, \ldots, a_{n-1})$ denote the amounts traded of each of the first $n-1$ assets, where $a_j > 0$ denotes a purchase, $a_j < 0$ a sale, and where $a_n = -\sum_{1}^{n-1} a_j$ for cash neutrality. Then this trading vector is given as the solution to:

$$\overline{A}\overline{a}' = P(i_0)\,[\overline{D}_0 - \overline{D}]^T \tag{11}$$

where $\overline{A}$ is given as in (7). Equation (11) may be readily solved for $\overline{a}'$ where $\overline{D}_0$ denotes the target total duration vector from (10A), and $P(i_0)$ and $\overline{D}$ denote the price and total duration vector of the original portfolio. A transpose symbol appears on the right-hand side of (11) to make the total duration vector into a column matrix.

Because of the constraints in (8), Equation (11) will always be solvable within the asset trading set. With enough assets, however, the solution need not be unique, so one will have significant latitude in choosing a trading solution based on other considerations.

AN EXAMPLE

Let $\bar{i}_0 = (0.075, 0.090, 0.100)$ represent semi-annual equivalent yields at six months, and five and ten years, respectively. The asset portfolio consists of \$50 million par of a ten-year, 12% coupon bond with market value of \$56.40 million and duration of 6.16, and \$17.48 million par of six-month commercial paper with a market value of \$16.85 million and duration of 0.48. Together, assets total \$73.25 million with a duration of 4.86. The single liability is a \$100 million GIC payment in year 5, with a market value of \$63.97 and duration of 4.86.

Consequently, surplus equals \$9.28 million, and the duration-matching strategy is chosen to immunize the surplus ratio of 12.67% against parallel yield curve shifts. The total duration vector for surplus is given by:

$$\overline{D}(i_0) = (4.20, -35.23, 35.88) \qquad (12)$$

We use monthly Treasury data from the period January 1987 through April 1994, from which is estimated the mean vector, $\overline{E} = E(\overline{\Delta i})$, and covariance matrix, $\overline{K}$:

$$\overline{E} = (-28.24, -62.23, -43.03)\,10^{-5}$$

$$\overline{K} = \begin{pmatrix} 0.63 & 0.71 & 0.58 \\ 0.71 & 1.21 & 1.05 \\ 0.58 & 1.05 & 0.97 \end{pmatrix} 10^{-5} \qquad (13)$$

In practice, because of the strong dependence of $\overline{E}$ on the economic cycle, it may be prudent to set $\overline{E}$ either equal to a forecast shift in which one wants to take a position, or, in the absence of such a forecast, implicitly equal to $\overline{0}$, and ignore it as a problem constraint. Below we develop returns using the $\overline{E}$ in Equation (13) for purposes of illustration only.

From Equation (4) with w = 1 and Equation (6), we easily calculate the monthly expected value and variance of $R(\overline{\Delta i})$:

$$E[R(\overline{\Delta i})] = 0.9947$$

$$\mathrm{Var}[R(\overline{\Delta i})] = 0.000773 \qquad (14)$$

Consequently, the current portfolio has an expected monthly return of –0.53%, and a monthly standard deviation of about 2.78%, based on data from this historical period.

RISK MINIMIZATION OF EXAMPLE

For the illustrations below, we consider two risk measures for minimization, RM (1) or the variance, and RM (0.99999), which puts about equal weight on the minimization of variance and worst case risk, because of the difference of scale between the units of $\overline{K}$ in (13), and those of the identity matrix $\bar{I}$. For convenience below, we refer to the corresponding minimization problems as Case 1 and Case 2. While RM (1) is given in (14), we note for comparisons below that RM (0.99999) = 0.026239 for the original portfolio, and $|\overline{D}| = 50.5$.

As a first application, we ignore asset trading sets and simply minimize risk in the portfolio subject to the one constraint of maintaining the current surplus duration of 4.85. That is, in (9) we have the one constraint: $\overline{D} \cdot \overline{N} = 4.85$, where $\overline{N} = (1, 1, 1)$. We then obtain from (10) the following:

$$\text{Case 1: } \overline{D}_0 = (5.20, -7.84, 7.48),\ \mathrm{RM} = 0.000099$$

$$\text{Case 2: } \overline{D}_0 = (2.35, 0.95, 1.55),\ \mathrm{RM} = 0.000262 \qquad (15)$$

In Case 1, the implied standard deviation is reduced 64%, from 2.78% to 0.99% per month, compared to the original portfolio. For Case 2, the variance reduction is 53% to 1.32% per month even though a reduction in standard deviation is not the primary objective of the problem. On a risk measure basis, the Case 1 reduction is 87%, while for Case 2, the risk measure is reduced 99% from its original value of 0.026239. The implied monthly returns on these risk-minimized portfolios are: –0.02% and +0.19%, respectively, while the $|\overline{D}|$ values are 12.0 and 3.0, respectively.

We next abandon the modified duration constraint in favor of a constraint on tradable assets. We consider three assets: six-month commercial paper; a

EXHIBIT 1
Asset Features

	Commercial Paper	5-Year Note	10-Year Bond
MV*	96.39	102.00	112.80
D_1	0.48	0.02	0.04
D_2	0	3.95	0.22
D_3	0	0	5.90
D	0.48	3.97	6.16

*Per 100 par.

five-year, 9 1/2 % coupon note; and the original ten-year, 12% coupon bond. Their relevant features are summarized in Exhibit 1.

First, we assume trading only between the five- and ten-year securities. The matrix $\overline{A} = [\overline{D}_1(\bar{i}_0) - \overline{D}_2(\bar{i}_0)]^T$ in (7) is given by:

$$\overline{A} = (0.02, -3.73, 5.90)^T \tag{16}$$

where $\overline{D}_1$ corresponds to the ten-year bond, $\overline{D}_2$ the five-year. Because the matrix $\overline{A}$ and its transpose clearly have rank equal to one, we expect two linearly independent solutions of (7). A calculation produces one such pair:

$$\overline{N}_1 = (0, 1.5818, 1), \qquad \overline{N}_2 = (-295, 0, 1) \tag{17}$$

Using these values and the original portfolio's total duration vector in (12), we obtain the constraint values from (8):

$$\overline{D} \cdot \overline{N}_1 = -19.8468$$

$$\overline{D} \cdot \overline{N}_2 = -1203.12$$

which when used in (10) produce the solutions:

$$\text{Case 1: } \overline{D}_0 = (4.16, -27.40, 23.50), \quad RM = 0.000575$$

$$\text{Case 2: } \overline{D}_0 = (4.07, -10.51, -3.23), \quad RM = 0.002866 \tag{18}$$

For this problem, we observe respective reductions in risk of 26% and 89% from the original portfolio, with corresponding changes in implied standard deviations of −14% and +39%. Respective expected monthly returns on the portfolio using (6) are −0.58% and −0.68%, while the respective duration values for these surplus portfolios have moved from 4.85 to 0.25 for Case 1, and to −9.67 for Case 2. Finally, respective values for $|\overline{D}|$ are 36.3 and 11.7.

To determine the necessary trades, we now solve (11), where here, $\overline{a}' = (a_1)$ denotes the trade of the ten-year bond, and $a_2 = -a_1$ equals the corresponding five-year bond trade. A calculation produces $a_1 = -9.48$ million in Case 1, and $a_1 = -61.52$ million in Case 2, both requiring a sale of ten-year assets and a purchase of the five-year.

As the last example, we solve the minimization problem with all three trading assets in Exhibit 1, both with and without a duration constraint. In both cases, we must solve (7) with:

$$\overline{A} = \begin{pmatrix} \overline{D}_1(\bar{i}_0) - \overline{D}_3(\bar{i}_0) \\ \overline{D}_2(\bar{i}_0) - \overline{D}_3(\bar{i}_0) \end{pmatrix}^T = \begin{pmatrix} -0.44 & -0.46 \\ 0.22 & 3.95 \\ 5.90 & 0 \end{pmatrix} \tag{19}$$

where $\overline{D}_i(\bar{i}_0)$, i = 1, 2, 3 denotes the total duration vectors of the ten-year bond, the five-year note, and the six-month commercial paper, respectively.

As $\overline{A}$ is easily seen to have rank equal to 2, there will be only one linearly independent solution to (7), for example:

$$N_1 = (1, 0.1165, 0.0702) \tag{20}$$

which has the corresponding constraint value from (8):

$$\overline{D} \cdot \overline{N}_1 = 2.6173$$

We first solve the minimization problem in (9) with only the above asset trading set constraint, and obtain:

$$\text{Case 1: } \overline{D}_0 = (2.84, -2.80, 1.38), \quad RM = 0.000016$$

$$\text{Case 2: } \overline{D}_0 = (2.70, -0.47, -0.40), \quad RM = 0.000100 \tag{21}$$

for respective risk reductions of 98% and 99.6%, respectively, corresponding reductions in implied monthly standard deviations of 86% and 83%, and expected monthly returns of −0.04% and +0.03%. Note that in this case the durations of both resulting portfolios again differ from the original value of 4.85, equaling 1.42 and 1.82, respectively. In addition, respective values of $|\overline{D}|$ are 4.2 and 2.8.

To fix the duration of the resulting surplus portfolio to equal 4.85, we add to the asset trading set constraint above the constraint $\overline{D} \cdot \overline{N} = 4.85$, with $\overline{N} = (1, 1, 1)$. This is not a problem in this case because $\overline{N}$ here is linearly independent from N_1 in (20) above.

Again solving (9), only with two constraints, we obtain:

$$\text{Case 1: } \overline{D}_0 = (2.79, -6.91, 8.97), \qquad RM = 0.000123$$

$$\text{Case 2: } \overline{D}_0 = (2.40, 0.93, 1.52), \qquad RM = 0.000262 \tag{22}$$

for respective risk reductions of 84% and 99%. These results are logically inferior to the risk reductions observed in (21) because of the presence of the additional constraint on the portfolio duration.

For these last two problems, we summarize trading results in Exhibit 2, obtained by solving (11) for $\overline{a'} = (a_1, a_2)$, and setting $a_3 = -a_1 - a_2$.

EFFICIENT FRONTIERS

Equation (10B) defines the risk, $RM_0(w)$, of the risk-minimizing portfolio as a function of $\overline{r}$, which quantifies the limitations on the constraints in (9), and of $\overline{B}$, which contains the direction vectors with respect to which these constraints are defined. Fixing these direction vectors, therefore, Equation (10B) can be thought of as describing an efficient frontier in (Risk, $\overline{r}$) space. It is efficient in the sense that for any given $\overline{r}$ value, and any portfolio that satisfies the constraints: $\overline{D} \cdot \overline{N}_j = r_j$, or, more compactly, $\overline{DB} = \overline{r}^T$, we have that the corresponding risk, RM (w), must satisfy:

$$RM\ (w) \geq RM_0(w) \tag{23}$$

For w = 1, the risk measure reduces to variance, so for the single direction vector, $\overline{N}_1 = \overline{E}$, the expected yield curve shift, Equation (10B) describes the risk/return efficient frontier of Markowitz, with a transformed return measure. This measure is transformed because by (6), the actual yield curve shift

EXHIBIT 2
Trading Results

	No-Duration Constraint		Duration = 4.85	
	Case 1	Case 2	Case 1	Case 2
10-year (a_1)	−54.27	−57.07	−42.33	−54.04
5-year (a_2)	79.20	84.83	68.89	87.96
6-month (a_3)	−24.94	−27.76	−26.56	−33.92

return for the portfolio is $r' = -r$.

In this simple two-dimensional model, we can rewrite (10B) to reflect an efficient frontier in true risk/return space, (RM, r′):

$$RM_0(1) = c\ (r')^2 \tag{24}$$

where c is the positive constant produced by the matrix product in (10B), with $\overline{B} = \overline{E}$. Clearly, this efficient frontier is a parabola in (RM, r′) space or (variance, return) space.

In the example, it is easy to solve the variance minimization problem for an arbitrary yield curve shift return specification, r′. To do this, we use (10) with $\overline{B} = \overline{E}$ as given in (13) to obtain:

$$\overline{D}_0 = (-1232.74, 4595.32, -3512.81)\ r'$$

$$\overline{D}_0 = 150.22\ r'$$

$$RM_0(1) = 16.307\ (r')^2 \tag{25}$$

For the given portfolio, r′ = −0.0053, D = 4.85, and variance = 0.000773. From (25), we see that the variance-minimizing portfolio with this return has $D_0 = -0.80$ and variance = 0.000458, for a 41% reduction.

As another simple application we can also investigate the efficient frontier in (variance, duration) space, or more generally, risk-duration space. Again using (10B), we obtain:

$$RM_0(1) = bD_N^2 \tag{26}$$

where b is the positive constant produced by the matrix product in (10B) with $\overline{B} = \overline{N}$, the direction vector of interest, and D_N denotes the targeted value for this directional duration. For the example, with $\overline{N} = (1, 1, 1)$ for traditional duration, we obtain:

$$\overline{D}_0 = (1.07, -1.62, 1.54)\, D_0$$

$$RM_0(1) = (4.193\, D_0^2)\, 10^{-6} \qquad (27)$$

Again, the efficient frontier in (26) and (27) is a parabola in (RM, D) space.

More generally, it can be shown that the efficient frontier in (10B) is always a paraboloid in (RM, $\bar{r}$) space. This is because the matrix $(\overline{B}^T \overline{K}_w^{-1} \overline{B})^{-1}$ is positive definite, as has been noted above.

In summary, by (10B) we have a general way of defining the efficient frontiers that result as a function of both portfolio returns and/or portfolio directional duration values. These directional durations may be explicitly desired, or implicitly required by (8) due to limitations imposed by the asset trading set. Of course, this frontier is also dependent on $\overline{B}$, which identifies the direction vectors used.

SUMMARY AND CONCLUSIONS

While traditional immunization is simple to implement, it often fails because the underlying assumption of parallel yield curve shifts is not realized. While formally generalizable to other models of yield curve movements, and even to a completely general model for these shifts, portfolio restrictions increase as more protection is sought. In the limit, only a cash-matching strategy, over yield curve regions, or more generally a dollar partial duration-matching strategy, will provide protection in the most general case.

To avoid the consequences of this conclusion, which may be unnecessarily restrictive because the theory may require protection even where it is not needed in practice, a new paradigm for immunization is required. We have introduced and illustrated the notion of stochastic immunization, which replaces the classical immunization paradigm of no downside risk with the more flexible paradigm of risk minimization.

Our examples illustrate solutions to the risk minimization problem when risk is defined to reflect a flexible balance of two important individual risk measures: variance and worst case risk. These problems can also be solved with restrictions reflecting a desired portfolio yield curve return, desired portfolio directional durations, and/or restrictions on the assets available for trading. All that is required is that the implied restrictions be consistent.

ENDNOTES

The author acknowledges the technical assistance of C. Dec Mullarkey and the support of the John Hancock in the development of the examples in this article.

[1]See also Theil [1971] and Wilks [1962].

[2]For more mathematical rigor and details on the theory, as well as additional considerations in its application, see Reitano [1993].

REFERENCES

Hiller, Randall S., and Christian Schaak. "A Classification of Structured Bond Portfolio Modeling Techniques." *Journal of Portfolio Management*, Fall 1990, pp. 37-48.

Ho, T.S.Y. "Key Rate Durations: Measures of Interest Rate Risks." *Journal of Fixed Income*, September 1992, pp. 29-44.

——. *Strategic Fixed Income Management*. Homewood, IL: Dow Jones-Irwin, 1990.

Litterman, Robert, and José Scheinkman. "Common Factors Affecting Bond Returns." *Journal of Fixed Income*, June 1991, pp. 54-61.

Markowitz, Harry. *Portfolio Selection: Efficient Diversification of Investments*. New York: John Wiley & Sons, 1959.

Martin, John D., Samuel H. Cox, and Richard D. MacMinn. *The Theory of Finance: Evidence and Applications*. New York: The Dryden Press, 1988.

Reitano, Robert R. "A Multivariate Approach to Duration Analysis." *Actuarial Research Clearing House*, Vol. 2 (1989), pp. 97-181.

——. "Non-Parallel Yield Curve Shifts and Durational Leverage." *Journal of Portfolio Management*, Summer 1990a, pp. 62-67.

——. "A Multivariate Approach to Immunization Theory." *Actuarial Research Clearing House*, Vol. 2 (1990b), pp. 261-312.

——. "Non-Parallel Yield Curve Shifts and Spread Leverage." *Journal of Portfolio Management*, Spring 1991a, pp. 82-87.

——. "Multivariate Duration Analysis." *Transactions of the Society of Actuaries*, XLIII (1991b), pp. 335-391.

——. "Multivariate Immunization Theory." *Transactions of the Society of Actuaries*, XLIII (1991c), pp. 392-438.

——. "Non-Parallel Yield Curve Shifts and Immunization." *Journal of Portfolio Management*, Spring 1992a, pp. 36-43.

——. "Non-Parallel Yield Curve Shifts and Convexity." *Transactions of the Society of Actuaries*, XLIV (1992b), pp. 479-507.

——. "Multivariate Stochastic Immunization Theory." *Transactions of the Society of Actuaries*, XLV (1993), pp. 425-484.

——. "Predictor Yield Curves and the Analysis of Period Yield Curve Shifts on Price." Working paper, 1995.

Theil, Henri. *Principles of Econometrics*. New York: John Wiley & Sons, 1971.

Wilks, Samuel. *Mathematical Statistics*. New York: John Wiley & Sons, 1962.

The Journal of Portfolio Management

VOLUME 22 NUMBER 2 WINTER 1996

A PUBLICATION OF INSTITUTIONAL INVESTOR, INC.