

# Contents

<b>Preface</b>	<b>ix</b>
<b>to Michael, David, and Jeffrey</b>	<b>xi</b>
<b>Introduction</b>	<b>xiii</b>
<b>1 Discussion of the Riemann Integral</b>	<b>1</b>
1.1 Weakness in the Definition of Existence . . . . .	1
1.2 Some Positive Results on Function Limits . . . . .	3
1.3 Counterexamples to Generalizing Positive Results . . . . .	7
<b>2 Lebesgue Integration</b>	<b>11</b>
2.1 Integrating Simple Functions . . . . .	11
2.1.1 Riemann Integral via Step Functions . . . . .	12
In 1-Dimension . . . . .	12
In $n$ -Dimensions . . . . .	16
2.1.2 Lebesgue Integral of Simple Functions . . . . .	20
2.2 Integrating More General Functions - A Discussion . . . . .	25
2.2.1 Riemann Integration . . . . .	25
Improper Intergals . . . . .	29
2.2.2 Lebesgue Integration . . . . .	29
2.3 Integrating Bounded Measurable Functions . . . . .	31
2.3.1 Definition of the Lebesgue Integral . . . . .	31
2.3.2 Riemann Implies Lebesgue on Rectangles . . . . .	34
2.3.3 Properties of the Lebesgue Integral . . . . .	36
2.3.4 Bounded Convergence Theorem . . . . .	39
2.4 Integrating Nonnegative Measurable Functions . . . . .	43
2.4.1 Definition and Properties of the Lebesgue Integral . . . . .	43
2.4.2 Integrating Nonnegative Function Sequences . . . . .	46

	Fatou's Lemma . . . . .	46
	Lebesgue's Monotone Convergence Theorem . . . . .	48
2.5	Integrating General Measurable Functions . . . . .	52
2.5.1	Definition and Properties of the Lebesgue Integral . . . . .	52
2.5.2	Lebesgue's Dominated Convergence Theorem . . . . .	56
2.5.3	Absolute Riemann Implies Lebesgue on $\mathbb{R}^n$ . . . . .	58
2.6	Summary of Convergence Results . . . . .	59
2.6.1	The Limit Theorems of Lebesgue Theory . . . . .	60
2.6.2	The Riemann Integral: A Discussion . . . . .	62
<b>3</b>	<b>Lebesgue Integration and Differentiation</b>	<b>69</b>
3.1	FTCs of the Riemann Theory . . . . .	69
3.2	Derivative of a Lebesgue Integral . . . . .	71
3.2.1	Monotonic Functions . . . . .	71
3.2.2	Functions Differentiable a.e. . . . .	86
3.2.3	Derivative of an Integral . . . . .	90
3.3	Lebesgue Integral of a Derivative . . . . .	96
3.3.1	Examples of FTC Failure . . . . .	96
3.3.2	The Lebesgue FTC - Version I . . . . .	105
3.4	Lebesgue Integration by Parts . . . . .	112
<b>4</b>	<b>Stieltjes Integration</b>	<b>115</b>
4.1	Riemann-Stieltjes Integrals on $\mathbb{R}$ . . . . .	117
4.1.1	Introduction and Definition . . . . .	117
4.1.2	Results on R-S Integrals . . . . .	122
	Existence of the R-S Integral . . . . .	122
	Properties of the R-S Integral . . . . .	132
	Integrators of Bounded Variation . . . . .	134
4.1.3	Evaluating Riemann-Stieltjes Integrals . . . . .	137
4.1.4	A Key on R-S Integration . . . . .	140
	Banach Spaces . . . . .	141
	Bounded Linear Functionals . . . . .	148
	A Key Result on R-S Integration . . . . .	154
4.2	Riemann-Stieltjes Integrals on $\mathbb{R}^n$ . . . . .	154
4.2.1	Introduction and Definition . . . . .	154
4.2.2	Results on R-S Integrals . . . . .	161
	Existence of R-S Integrals . . . . .	161
	Properties of R-S Integrals . . . . .	164
4.2.3	Evaluating Riemann-Stieltjes Integrals . . . . .	167

**CONTENTS**

vii

**References**

**171**