

Contents

Preface	ix
to Lisa	xi
Introduction	xiii
1 Stochastic Calculus: An Informal Link to Finance	1
1.1 Asset Model Limits	1
1.2 Outline of Stochastic Calculus Topics	6
1.2.1 Book 8	6
1.2.2 Book 9 or Later	7
2 The Itô Integral	9
2.1 Is a New Integration Theory Needed?	9
2.2 Brownian Motion	12
2.2.1 Additional Properties of BM	14
2.2.2 BM on a Filtered Probability Space	18
2.3 Preliminary Insights to a New Integral	22
2.3.1 Axioms for a New Integral	23
2.3.2 Next Steps and Questions	25
2.4 Quadratic Variation Process of Brownian Motion	27
2.5 Itô Integral of Simple Processes	32
2.6 $H_2([0, \infty) \times \mathcal{S})$ and Simple Process Approximations	39
2.7 The General Itô Integral	48
2.8 Properties of the Itô Integral	54
2.9 A Continuous Version of the Itô Integral	58
2.10 Itô Integration via Riemann Sums	65

3	Integrals w.r.t. Continuous Local Martingales	79
3.1	Integration of Simple Processes	81
3.2	Integrals w.r.t. Continuous L_2 -Bounded Martingales	84
3.2.1	A Generalized Itô Isometry	84
3.2.2	\mathcal{M}^2 -Integrators and $H_2^M([0, \infty) \times \mathcal{S})$ -Integrands	89
3.2.3	Simple Process Approximations in $H_2^M([0, \infty) \times \mathcal{S})$	94
3.2.4	The General Stochastic Integral	99
3.2.5	A Continuous Version	106
3.2.6	The Kunita-Watanabe Inequality	109
	Measures Induced by BV Functions	110
	The Kunita-Watanabe Inequality	117
3.2.7	Additional Properties For L_2 -Bounded Martingales	124
3.2.8	Stochastic Integrals via Riemann Sums	137
3.3	Integrals w.r.t. Continuous Local Martingales	142
3.3.1	\mathcal{M}_{loc} -Integrators and $H_{2,loc}^M([0, \infty) \times \mathcal{S})$ -Integrands	148
3.3.2	The General Stochastic Integral	155
3.3.3	Properties of Stochastic Integrals	161
3.3.4	Stochastic Dominated Convergence Theorem	175
3.3.5	Stochastic Integrals via Riemann Sums	177
4	Integrals w.r.t. Continuous Semimartingales	181
4.1	Integrals w.r.t. Continuous B.V. Processes	185
4.2	The General Stochastic Integral	195
4.3	Properties of Stochastic Integrals	198
4.4	Stochastic Dominated Convergence Theorem	201
4.5	Stochastic Integrals via Riemann Sums	203
4.6	Stochastic Integration by Parts	204
4.7	Integration of Vector and Matrix Processes	207
4.7.1	The Itô Integral	209
4.7.2	Integrals w.r.t. Continuous Semimartingales	212
5	Itô's Lemma	215
5.1	Versions of Itô's Lemma	218
5.2	Semimartingale Version	219
5.3	Itô Process Version	227
5.4	Itô Diffusion Version	228
5.5	Multivariate Semimartingale Version	237
5.6	Multivariate Itô Process Version	244
5.7	Multivariate Itô Diffusion Version	247

6	Some Applications of Itô's Lemma	255
6.1	Lévy's Characterization of n -Dimensional BM	255
6.1.1	When is an Itô Process a Brownian Motion?	259
6.1.2	Continuous LMs are Time-Changed BMs	267
6.2	The Burkholder-Davis-Gundy Inequality	276
6.3	Local Martingales from Semimartingales	284
6.3.1	Itô Diffusion Version	284
6.3.2	Semimartingale Version	289
6.4	The Feynman-Kac Representation Theorem 1	293
6.5	Dynkin's Formula	301
6.5.1	Infinitesimal Generators	309
6.5.2	Kolmogorov's Backward Equation 1	311
	References	321